THE CENTROID OF AREAS AND VOLUMES.

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It is the object of this paper to put on record, once for all, general values for the centroid of areas, represented by the curve $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the centroid of volumes represented by the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1.$$

I. *Areas.* Let the density vary as $x^{k-1}y^{l-1}$, the thickness being constant.

$$\text{Then } \bar{x}=\frac{\iint x^k y^{l-1}dxdy}{\iint x^{k-1}y^{l-1}dxdy},\quad \bar{y}=\frac{\iint x^{k-1}y^{l}dxdy}{\iint x^{k-1}y^{l-1}dxdy}.$$

$$\therefore\ \bar{x}=\frac{\dfrac{a^{k+1}b^l}{\dfrac{4}{(2m+1)(2n+1)}}\ \dfrac{\Gamma\left\{\frac{k+1}{2}(2m+1)\right\}\Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k+1}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}{\dfrac{a^k b^l}{\dfrac{4}{(2m+1)(2n+1)}}\ \dfrac{\Gamma\left\{\frac{k}{2}(2m+1)\right\}\Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}.$$

$$\therefore\ \bar{x}=\frac{\Gamma(km+m+\frac{k+1}{2})\ \Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(km+\frac{k}{2})\Gamma(km+ln+m+\frac{k+l+1}{2}+1)}a \quad \ldots\ldots\ldots\ldots (A).$$

$$\text{Similarly, } \bar{y}=\frac{\Gamma(ln+n+\frac{l+1}{2})\Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(ln+\frac{l}{2})\Gamma(km+ln+n+\frac{k+l+1}{2}+1)}b \quad \ldots\ldots\ldots\ldots (B).$$

This gives the centroid of a quadrant of the area whatever be the values of k, l, m, n. Let $k=l=1$, so that the density is the same throughout the whole area.

$$\therefore \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+n+2)}{\Gamma(m+\frac{1}{2})\Gamma(2m+n+\frac{5}{2})}a,\ \bar{y}=\frac{\Gamma(2n+1)\Gamma(m+n+2)}{\Gamma(n+\frac{1}{2})\Gamma(m+2n+\frac{5}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore \bar{x}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}a=\frac{4a}{3\pi},\ \bar{y}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}b=\frac{4b}{3\pi}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore \bar{x}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}a=\frac{256a}{315\pi},\ \bar{y}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}b=\frac{256b}{315\pi}.$$

Let $m=n=2$, then $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1$.

$$\therefore \bar{x}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}a=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4a}{\pi},$$

$$\bar{y}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}b=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4b}{\pi}.$$

Let $m=0, n=1$, then $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore \bar{x}=\frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}a=\frac{16a}{15\pi},\ \bar{y}=\frac{\Gamma(3)\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{9}{2})}b=\frac{128b}{105\pi}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

$\therefore \bar{x}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}a=\frac{a}{5},\ \bar{y}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}b=\frac{b}{5}$, the centroid of the area between the parabola and its tangents as axes.

Let the density vary as xy, so that $k=l=2$.

$$\therefore \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2m+1)\Gamma(3m+2n+\frac{7}{2})}a,\ \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2n+1)\Gamma(2m+3n+\frac{7}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}a=\frac{8a}{15},\ \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}b=\frac{8b}{15}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}a=\frac{128a}{429},\ \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}b=\frac{128b}{429}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}a=\frac{2a}{9},\ \bar{y}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}b=\frac{2b}{9}.$$

Let the density vary as x the distance from the axis of ordinates so that $k=2,\ l=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+n+\frac{5}{2})}{\Gamma(2m+1)\Gamma(3m+n+3)}a,\ \bar{y}=\frac{\Gamma(2n+1)\Gamma(2m+n+\frac{5}{2})}{\Gamma(n+\frac{1}{2})\Gamma(2m+2n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{b}\right)^{2}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}a=\frac{3\pi a}{16},\ \bar{y}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}b=\frac{3b}{8}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}a=\frac{49.45\pi a}{2^{14}},\ \bar{y}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}b=\frac{63b}{384}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}a=\frac{5a}{14},\ \bar{y}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}b=\frac{3b}{28}.$$

Let the density vary as y the distance from the axis of abscissas so that $k=1,\ l=2$.

$$\therefore\ \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+2n+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(2m+2n+3)}a,\ \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(m+2n+\frac{5}{2})}{\Gamma(2n+1)\Gamma(m+3n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}a=\frac{3a}{8},\ \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}b=\frac{3\pi b}{16}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}a=\frac{63a}{384},\ \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}b=\frac{49.45\pi b}{2^{14}}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

$$\therefore\ \bar{x}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}a=\frac{3a}{28},\ \bar{y}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}b=\frac{5b}{14}.$$

[To be Continued.]